6m

(a) $\hat{A} = \frac{d}{dx}$ $\quad \hat{E} = x^2$

$$\hat{A}^2 f(x) = \hat{A}[\hat{A} f(x)] = \hat{A}\left[\frac{d}{dx} f(x)\right]$$

$$= \frac{d}{dx}\left[\frac{d}{dx} f(x)\right] = \frac{d^2}{dx^2} f(x)$$

$$\therefore \hat{A}^2 \equiv \frac{d^2}{dx^2}$$

$$[\hat{A} f(x)]^2 = \left(\frac{d}{dx} f(x)\right]^2 \neq \hat{A}^2 f(x)$$

(b) $\hat{A}\hat{E} f(x) = \frac{d}{dx}[x^2 f(x)]$

$$= 2x f(x) + x^2 \frac{d f(x)}{dx}$$

$$\hat{E}\hat{A} f(x) = x^2 \left[\frac{d}{dx} f(x)\right]$$

$$= x^2 \frac{d f(x)}{dx}$$

hence $\quad \hat{A}\hat{E} f(x) \neq \hat{E}\hat{A} f(x)$

7. (a) e^{ikx} $\Rightarrow$ eigenfns of both $\frac{d}{dx}$ and $\frac{d^2}{dx^2}$

$$\frac{d}{dx} e^{ikx} = \underbrace{(ik)}_{\text{eigenvalue}} e^{ikx}$$

$$\frac{d^2}{dx^2} e^{ikx} = (ik)^2 e^{ikx} = \underbrace{-k^2}_{\text{eigen value}} e^{ikx}$$

(b) $$\frac{d}{dx} \cos kx = -k \underbrace{\sin kx}_{\text{not the original function}}$$

$\cos kx$ is not an eigenfunction of the $\frac{d}{dx}$

$$\frac{d^2}{dx^2} \cos kx = \underbrace{-k^2}_{\text{eigenvalue}} \underbrace{\cos kx}_{\text{original fn. comes back}}$$

(c) $$\frac{d}{dx} k = 0 = \underbrace{0}_{\text{eigenvalue}} . k$$

$$\frac{d^2}{dx^2} k = 0 \qquad \text{eigenvalue} = 0$$

7 (d) $\frac{d}{dx}(kx) = k = \underbrace{\frac{1}{x}}_{\text{not a constant; hence cannot be an eigenvalue}}(kx)$ ← original fn.

kx is not an eigenfn. of $\frac{d}{dx}$

$\frac{d^2}{dx^2}(kx) = 0 = \underbrace{0}_{\text{eigenvalue}}.(kx)$

kx is an eigenfn. of $\frac{d^2}{dx^2}$

(e) $e^{-ax^2} = f(x)$

$\frac{d}{dx}(e^{-ax^2}) = \underbrace{-2ax}_{\text{not a constant; hence not an eigenvalue}} \overbrace{(e^{-ax^2})}^{\text{not an eigenfn. of } \frac{d}{dx}}$

$\frac{d^2}{dx^2}(e^{-ax^2}) = \frac{d}{dx}\left\{-2ax(e^{-ax^2})\right\}$

$= \underbrace{(-2a + 4a^2x^2)}_{\text{not an eigenvalue}} e^{-ax^2}$

e^{-ax^2} is not an eigenfn. of $\frac{d^2}{dx^2}$

8.

- $$\frac{d^2}{dx^2}\cos \omega x = \underline{\underline{-\omega^2}} \cos \omega x$$

- $$\frac{d}{dt} e^{i\omega t} = \underline{\underline{i\omega}}\, e^{i\omega t}$$

- $$\left(\frac{d^2}{dx^2} + 2\frac{d}{dx} + 3\right) e^{\alpha x}$$

  $$= \underline{\underline{(\alpha^2 + 2\alpha + 3)\, e^{\alpha x}}}$$

- $$\frac{\partial}{\partial y} x^2 \exp(6y) = \underline{\underline{6}}\, x^2 \exp(6y)$$

9.

$$\nabla^2 (\cos ax)(\cos by)(\cos cz)$$

$$= \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right] \left\{ (\cos ax)(\cos by)(\cos cz) \right\}$$

$$= -a^2 (\cos ax)(\cos by)(\cos cz) - b^2 (\cos ax)(\cos by)(\cos cz)$$
$$- c^2 (\cos ax)(\cos by)(\cos cz)$$

$$= \underline{\underline{-(a^2+b^2+c^2)}} (\cos ax)(\cos by)(\cos cz)$$

eigen value

15.

(a) $$\int_0^L N \sin\frac{n\pi x}{L} \, \underbrace{N \sin\frac{n\pi x}{L}} \, dx = 1$$ wave fn. for a particle in a 1-D box

$$\therefore N^2 \int_0^L \sin^2 \frac{n\pi x}{L} \, dx = 1$$

[using ~~cos 2x = ...~~ $\cos 2x = 1 - 2\sin^2 x$

$$\therefore \sin^2 x = \frac{1}{2}(1 - \cos 2x) \Big]$$

$$\therefore \frac{N^2}{2} \int_0^L \left[1 - \cos\frac{2n\pi x}{L}\right] dx = 1$$

$$\frac{N^2}{2}\left[\int_0^L dx - \int_0^L \cos\frac{2n\pi x}{L} \, dx\right] = 1$$

$$\therefore \frac{N^2}{2}\left[L - \left\{\frac{L}{2n\pi} \sin\frac{2n\pi x}{L}\right\}_0^L\right] = 1$$

$$\therefore \frac{N^2}{2}\left[L - \frac{L}{2n\pi}\underbrace{\left\{\sin 2n\pi\right\}}_{0}\right] = 1$$

$$\therefore \frac{N^2}{2} L = 1$$

$$\text{or} \quad \underbrace{N = \sqrt{\frac{2}{L}}}$$

(b) $$N^2 \int_{-L}^{L} c^2 \, dx = 1$$

$$N^2 c^2 \int_{-L}^{L} dx = 1$$

$$\therefore \quad 2L N^2 c^2 = 1$$

$$\therefore \quad N = \frac{1}{c\sqrt{2L}}$$

(c) Here you will have to use spherical polar coordinates.

$$\underbrace{dx\,dy\,dz}_{\substack{\downarrow \\ \text{cartesian} \\ \text{coordinates}}} = d\tau = \underbrace{r^2 \sin\theta \, dr \, d\theta \, d\phi}_{\substack{\downarrow \\ \text{spherical polar coordinates}}}$$

$$N^2 \int_0^{\infty} r^2 e^{-r/a_0} \cdot e^{-r/a_0} \, dr \int_0^{\pi} \sin\theta \, d\theta \int_0^{2\pi} d\phi = 1$$

$$\therefore \quad N^2 \int_0^{\infty} r^2 e^{-2r/a_0} \, dr \cdot [2] \cdot [2\pi] = 1$$

$$\therefore \quad 4\pi N^2 \underbrace{\int_0^{\infty} r^2 e^{-2r/a_0} \, dr}_{\text{standard integral given with the question}} = 1$$

$$\left[\int_0^{\infty} x^n e^{-ax} dx = \frac{n!}{a^{n+1}} \right]$$

$$\therefore \quad 4\pi N^2 \left(\frac{a_0^3}{4} \right) = 1 \qquad \therefore \quad N = \frac{1}{(\pi a_0^3)^{1/2}}$$

(d) $x \exp(-r/2a_0)$ in 3D

$x = r\sin\theta\cos\phi$

$\therefore N^2\psi^2 = N^2 \; r^2\sin^2\theta\cos^2\phi \; e^{-r/a_0}$

$$\therefore \; N^2\int_0^\infty r^2 . r^2 e^{-r/a_0}\, dr \int_0^\pi \sin^3\theta \, d\theta \int_0^{2\pi} \cos^2\phi \, d\phi = 1$$

Do this integral by yourself

Answer: $N = \dfrac{1}{(32\pi a^5)^{1/2}}$

17. (a) $e^{-x^2/2}$ $(-\infty, \infty)$

both at $-\infty$ and $+\infty$ $e^{-x^2/2}$ converse; hence this is a valid wavefn and so is normalizable.

<u>To normalize</u>

$$\int \psi^* \psi \, dx = 1$$

or $$\int_{-\infty}^{+\infty} N^* e^{-x^2/2} \, N e^{-x^2/2} = 1$$

or $$|N|^2 \int_{-\infty}^{+\infty} e^{-x^2/2} = 1$$

or $$2|N|^2 \underbrace{\int_0^{\infty} e^{-x^2/2}}_{\text{standard integral}} = 1$$ [$e^{-x^2/2}$ is an odd function]

or $$2|N|^2 \left(\frac{\pi}{4}\right)^{1/2} = 1$$

$$\therefore \; |N|^2 . \pi^{1/2} = 1$$

$$\therefore \; |N|^2 = \frac{1}{\pi^{1/2}}$$

$$\therefore \; N = \frac{1}{\pi^{1/4}}$$ [neglecting -ve sign]

17 (c) $e^{i\theta}$ $(0, 2\pi)$ → can be normalized

$$N^2 \int_0^{2\pi} e^{-i\theta} e^{i\theta} \, d\theta = 1$$

$$\therefore N = \left(\frac{1}{2\pi}\right)^{1/2}$$

(e) $$N^2 \int_0^{\infty} dx \int_0^{\infty} dy \; e^{-(x^2+y^2)/2} \, e^{-(x^2+y^2)/2}$$

$$\therefore N^2 \int_0^{\infty} e^{-x^2} dx \int_0^{\infty} e^{-y^2} dy = 1$$

or $$N^2 \left(\frac{\pi}{4}\right)^{1/2} \left(\frac{\pi}{4}\right)^{1/2} = 1$$

$$\therefore N^2 = \frac{4}{\pi}$$

and $$N = \frac{2}{\sqrt{\pi}}$$

wavefn. converges at the limits of 0 and ∞, hence it would be normalized

(b) e^{x} $(0, \infty)$

the function diverges at $x \to \infty$

hence e^{x} cannot be normalized

(d) xe^{x} $(0, \infty)$

here also the function diverges $\to$ cannot be normalized

however if the function had been

xe^{-x} $(0, \infty)$

then it would have been normalizable

18. $\psi(x) = \overset{c_1}{\frac{1}{2}}\phi_1(x) + \overset{c_2}{\frac{1}{4}}\phi_2(x) + \overset{c_3}{\frac{3+i\sqrt{2}}{4}}\phi_3(x)$

$\phi_1(x)$, $\phi_2(x)$ and $\phi_3(x)$ are normalized

given

considering $\hat{H}$ to be the K.E. operator

$$\hat{H}\,\phi_1(x) = E_1\,\phi_1(x)$$
$$\hat{H}\,\phi_2(x) = 3E_1\,\phi_2(x)$$
$$\hat{H}\,\phi_3(x) = 7E_1\,\phi_3(x)$$

(b) these are the 3 possible values for K.E.

(a) $\int_{\text{all space}} \psi^* \psi \, dx = |c_1|^2 + |c_2|^2 + |c_3|^2$

$= \frac{1}{4} + \frac{1}{16} + \frac{11}{16} = \frac{16}{16} = 1$

Hence $\psi(x)$ is also normalized

(d) $\langle E \rangle = \dfrac{\int \psi^* \hat{H} \psi \, dx}{\int \psi^* \psi \, dx} = 1$ [found above]

$= \int \psi^* \hat{H} \psi$

$= |c_1|^2 E_1 + |c_2|^2 3E_1 + |c_3|^2 7E_1$

respective eigen values

$= \frac{E_1}{4} + \frac{3E_1}{16} + \frac{11}{16} \cdot 7E_1$

$= \frac{84E_1}{16} = \underline{\underline{5.25 E_1}}$

(c) $P = |c_K|^2$ i.e. $\frac{1}{4}$ for (E_1); $\frac{1}{16}$ for $(3E_1)$; $\frac{11}{16}$ for $(7E_1)$